ANALYSIS OF THE FALKNER–SKAN EQUATION IN BOUNDARY LAYERS AND ITS HYDRODYNAMIC APPLICATIONS

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1. Introduction

One of the fundamental problems of modern hydro- and aerodynamics is the study of the motion of a viscous fluid along the surface of solid bodies. The boundary-layer theory proposed in 1904 by the German scientist Ludwig Prandtl brought a revolutionary change to fluid mechanics [1]. This theory makes it possible, at high Reynolds numbers, to decompose the flow into two distinct regions: a thin layer adjacent to the surface where viscous (frictional) forces are dominant, and an outer potential flow region where the effects of viscosity are negligible.

Initially, G. Blasius (1908) presented his classical solution for the flow over a flat plate [2]. However, in real engineering applications (such as aircraft wings and turbine blades), the surface is rarely perfectly flat, and the external flow velocity is generally not constant. The presence of a pressure gradient necessitates a more general description of the boundary layer. In 1931, the equation proposed by V. M. Falkner and S. W. Skan addressed this issue by providing a similarity-based model for flows in accelerating or decelerating external velocity fields [3].

2. Physical and Mathematical Formulation of the Problem

The Falkner–Skan problem concerns the laminar boundary layer developing over a wedge-shaped body, either converging or diverging. The external flow velocity $U(x)$ is assumed to vary with the surface coordinate x according to a power-law distribution [4]:

$$U(x) = Cx^m,$$

where C is a constant and m is a parameter determined by the geometry of the flow.

The opening angle of the converging or diverging body is equal to $\pi\beta$, where β (the Hartree parameter) is defined by the relation

$$\beta = \frac{2m}{m+1}$$

This formulation allows the influence of favorable or adverse pressure gradients on the boundary-layer structure to be systematically analyzed through the single parameter β.

The Navier–Stokes equations are simplified and, by introducing the stream function ψ, the resulting system of partial differential equations is reduced to a single ordinary differential equation (ODE). For this purpose, the following similarity variable η and the dimensionless stream function $f(\eta)$ are used [5]:

$$\eta = y\sqrt{\frac{U(x)(m+1)}{2\nu x}}, \quad \psi(x,y) = \sqrt{\frac{2\nu x U(x)}{m+1}} f(\eta).$$

As a result, the Falkner–Skan equation is obtained:

$$f''' + ff'' + \beta(1-(f')^2) = 0.$$

To solve the problem, the following physical boundary conditions are imposed [1,5]:

1. No-slip condition at the wall **(** $y=0$ **):** the fluid adheres to the surface

$$f(0)=0, \quad f'(0)=0.$$

2. Condition at infinity **(** $y \to \infty$ **):** the velocity approaches the external flow

$$f'(\infty)=1.$$

3. Parameter analysis and flow regimes

In the Falkner–Skan equation, the parameter β is the main factor that determines the physical nature of the flow.

$\beta = 0$ (zero pressure gradient): In this case, the equation reduces to the Blasius equation ($f''' + 0.5ff'' = 0$). This corresponds to the flow over a flat plate.

$\beta > 0$ (favorable pressure gradient): The flow accelerates ($dp/dx < 0$). The case $\beta = 1$ is referred to as a stagnation-point flow. In this regime, the boundary layer becomes thinner and more stable [4].

$\beta < 0$ (adverse pressure gradient): The flow decelerates ($dp/dx > 0$). The most important critical value is $\beta \approx -0.1988$. At this point, the velocity gradient at the wall becomes zero ($f''(0) = 0$), meaning that the shear stress vanishes and flow separation from the surface occurs [6].

4. Numerical modeling and results

Since the Falkner–Skan equation is strongly nonlinear, an exact analytical solution does not exist. Therefore, the shooting method combined with Runge–Kutta algorithms was used to solve it numerically [6].

The graphs obtained using a Python program for different values of the parameter β, which compute the velocity profiles (Figure 1), demonstrate the following:

Deformation of the velocity profile: as the absolute value of the parameter β increases (from -0.1 to -0.1988), the velocity profile gradually becomes "flatter." This behavior is explained by the deceleration of the external flow and the associated decrease in the kinetic energy of the fluid within the boundary layer.

Decrease in shear stress: the slope of the curve at the origin of the coordinate system ($y = 0$ or $\eta = 0$) is directly proportional to the wall shear stress on the surface ($\tau_w \sim f''(0)$).

As shown in the figure, a decrease in β leads to a reduction in the initial curvature of the profile. Separation point: the lowest curve ($\beta = -0.1988$) is of particular importance. In this case, the derivative of the velocity profile at the wall becomes zero:

$f''(0) \approx 0$,

indicating vanishing shear stress and the onset of flow separation.

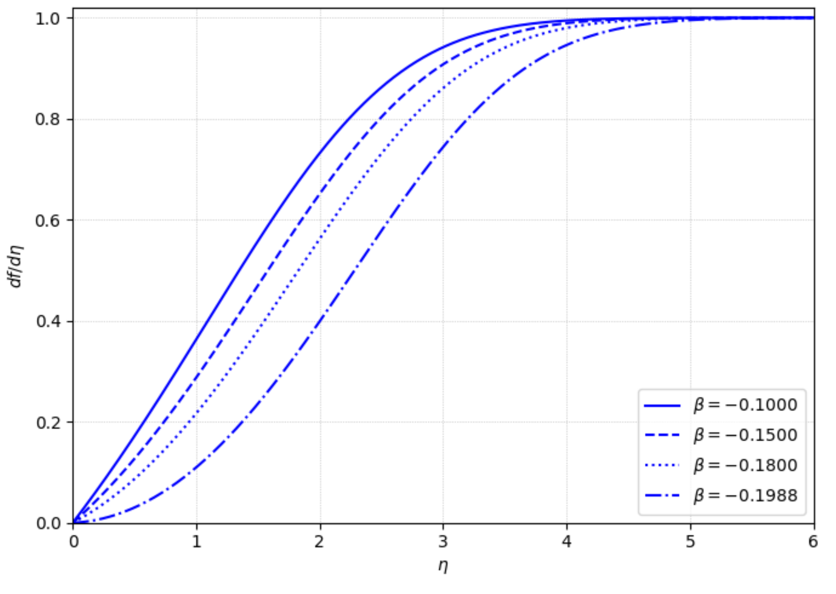

Figure 1. Velocity profiles for the Falkner–Skan flow under adverse pressure gradients (β<0)

The graph illustrates the deformation of the velocity profile as the parameter β decreases from -0.1000 to -0.1988, as well as the approach to the boundary-layer separation state at the critical value $\beta_{\text{crit}} \approx -0.1988$.

5. Conclusion

In this study, the motion of a viscous fluid over converging or diverging surfaces was investigated using the Falkner–Skan equation. The results of the numerical modeling show that both the sign and the magnitude of the pressure gradient have a decisive influence on boundary-layer stability. A favorable pressure gradient ($\beta > 0$) stabilizes the flow, whereas an adverse pressure gradient ($\beta < 0$) leads to flow separation from the surface.

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