SPECTRAL-GRID APPROXIMATION OF A BOUNDARY VALUED PROBLEM FOR A FOURTH-ORDER SINGULARLY PERTURBED EQUATION

Murodov Sardorbek
Termez State University
smurodov870@gmail.com

Abstract

Many studies devoted to solving boundary value problems for singularly perturbed differential equations propose the use of nonuniform and adaptive meshes in order to localize the boundary layer. However, the construction of such meshes generally requires the use of specific mappings, whose design can be associated with considerable difficulties and may lead to a significant increase in the number of arithmetic operations. In this paper, the numerical modeling of a boundary value problem for a fourth-order singularly perturbed differential equation using the spectral-grid method is considered. First, the convergence of the spectral-grid method is theoretically justified, and estimates for the rate of convergence are obtained. Then, the considered problem is approximated using the spectral-grid method, and a corresponding solution algorithm is developed.

Keywords: singularly perturbed boundary value problem; spectral-grid method; Chebyshev polynomial; maximum absolute error; numerical modeling

1.Introduction

For singularly perturbed boundary value problems of this type, the development of efficient numerical methods that provide high accuracy and parameter-uniform convergence is one of the important research directions of modern computational mathematics. In particular, the presence of a small parameter ε often leads to the formation of boundary layers, which significantly reduces the efficiency of approximations based on classical difference schemes[1]. Motivated by this issue, the present work develops a spectral-grid method based on the first-kink Chebyshev polynomials for the numerical solution of singularly perturbed boundary value problems with multiple boundary layers. The proposed method combines non-uniform domain decomposition with independently selected polynomial degrees on each subdomain, providing the flexibility to employ finer discretization and higher polynomial approximations in boundary-layer regions while using courser discretization in smooth regions. As a result, the method achieves and effective balance between computational efficiency and approximation accuracy.

2. Statement of the problem and solution method

Consider the following fourth-order singularly perturbed boundary value problem with a small parameter at the derivative

$$\varepsilon u''''(x) - u'''(x) - u''(x) + u'(x) + u(x) = f(x), x \in [0,1] \qquad (1)$$

subject to the boundary conditions

$$u(0) = u(1) = u'(0) = u''(1) = 0, \qquad (2)$$

Here ε is a small positive parameter. To numerically model the differential problem (1)-(2), we employ the spectral-grid method based on Chebyshev polynomials of the first kind as basis functions. In order to investigate the convergence and accuracy of the proposed numerical method, the method of test functions is applied. According to this approach, an arbitrary function $u_e(x)$ satisfying the boundary conditions (2) is chosen. Substituting this function into equation (1), the corresponding right-hand side function $f(x)$ is determined. The considered problem is then solved by the spectral-grid method, and the obtained approximate solution is compared with the chosen test function.

To analyze problem (1)-(2), the following function satisfying the boundary conditions (2) is chosen as a test function:

$$u_e(x) = c_1 x^3 + c_2 x^2 + c_3 x + \frac{e^{-1/\varepsilon} - e^{-(1-x)/\varepsilon}}{1 - e^{-1/\varepsilon}} \varepsilon^2, \qquad (3)$$

where

$$c_1 = -\frac{\varepsilon^2}{2} + \frac{\varepsilon e^{-1/\varepsilon}}{2(1 - e^{-1/\varepsilon})} + \frac{1}{4(1 - e^{-1/\varepsilon})},$$

$$c_2 = \frac{3}{2}\varepsilon^2 - \frac{3\varepsilon e^{-1/\varepsilon}}{2(1 - e^{-1/\varepsilon})} - \frac{1}{4(1 - e^{-1/\varepsilon})}, \quad c_3 = \frac{\varepsilon e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

The spectral-grid method is constructed according to the following algorithm [2]. First, the computational domain $[a,b]$ is divided into M non-overlapping subintervals

$$a = x_0 < x_1 < \cdots < x_M = b.$$

On each subinterval $I_j = [x_{j-1}, x_j]$, $j = 1, \ldots, M$, the unknown solution is approximated by a finite expansion in the first-kind Chebyshev polynomials:

$$u_j(x) = \sum_{k=0}^{N_j} c_{j,k} T_k(\xi_j(x)) ,$$

where N_j is the polynomial degree on the j-th subinterval, $c_{j,k}$ are unknown coefficients, and $\xi_j(x)$ is the linear transformation of I_j onto the standard interval [-1,1]. The Chebyshev points are then introduced on each subinterval and mapped from [-1,1] to I_j. Using the corresponding differentiation matrices, the derivatives appearing in the differential equations are expressed in terms of the unknown coefficients. The resulting spectral representations are substituted into the differential equation at the corresponding collocation points, producing a local algebraic system for each subinterval.

The boundary conditions are imposed at the endpoints of the computational domain. At each internal interface $x = x_j$, the required continuity conditions for the solution and its derivatives are imposed. Consequently, the local algebraic system is combined into a global system of linear algebraic equations for the unknown coefficients $c_{j,k}$. After solving this system, the coefficients of the Chebyshev expansions are obtained and the numerical solution is reconstructed on each subinterval.

3. Discussion of Results

Table 1. Maximum absolute error of the spectral-grid method consisting of two grid elements ($N = 2$).

ε	Total number of polynomials $M = \sum_{j=1}^{N}(p_j + 1)$	Spectral method Δ	Spectral-grid method Δ			
			$p_1 + 1$	$p_2 + 1$	Lengths of grid elements	
					[0, 0.9], [0.9, 1]	[0, 0.98], [0.98, 1]
10^{-4}	64	9.1×10^{-3}	32	32	6.3×10^{-4}	3.6×10^{-5}
			16	48	5.4×10^{-4}	3.1×10^{-7}
			8	56	1.2×10^{-3}	1.2×10^{-7}
	128	6.1×10^{-5}	64	64	3.0×10^{-6}	1.5×10^{-13}
			32	96	9.3×10^{-8}	1.0×10^{-22}
			18	114	9.3×10^{-9}	4.7×10^{-29}
	256	2.8×10^{-8}	128	128	3.2×10^{-12}	2.1×10^{-38}
			64	192	1.5×10^{-21}	5.4×10^{-72}
			32	224	4.2×10^{-26}	5.9×10^{-89}

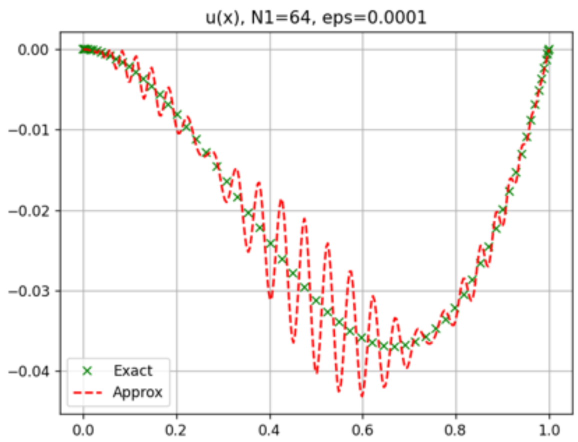

Figure 1. Spectral-grid method

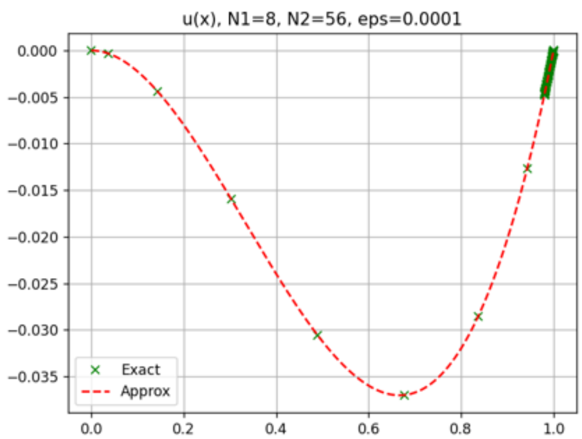

Figure 2. Spectral method

From the results in Table 1 and the Figures it can be observed that increasing the number of grid elements and reducing the size of the grid elements near the boundary leads to a significant clustering of Chebyshev nodes. This clustering results in a substantial reduction of the maximum absolute error. The use of the spectral-grid method allows the number and lengths of the grid elements to be selected according to the spectral features of the problem under consideration. For example, in the fourth-order singularly perturbed problem studied here, when the parameter ε takes very small values, the solution exhibits sharp variations near the boundary regions. In such cases, if the number and lengths of the grid elements are not chosen appropriately, sufficient accuracy cannot be achieved even when the total number of Chebyshev polynomials M is large.

References

[1] Normurodov Ch.B., Murodov S.K., Shakaeva E.E. Spectral-grid approximation of an ordinary differential equation with a small parameter at the highest derivative //Problems of Computational and Applied Mathematics. -2025. -Vol. 2(64). -P. 54-63. -DOI:10.71310/pcam.2_64.2025.05.

[2] Normurodov, Ch.B., Tursunova, B.A. Numerical modeling of the boundary value problem of an ordinary differential equation with a small parameter at the highest derivative by Chebyshev polynomials of the second kind. Results in Applied Mathematics. 19 (2023), 100388, https://doi.org/10.1016/j.rinam.2023.100388.